

# A real sextic surface with 45 handles

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## Abstract

It follows from classical restrictions on the topology of real algebraic varieties that the first Betti number of the real part of a real nonsingular sextic in  $\mathbb{CP}^3$  can not exceed 94. We construct a real nonsingular sextic  $X$  in  $\mathbb{CP}^3$  satisfying  $b_1(\mathbb{R}X) = 90$ , improving a result of F.Bihan. The construction uses Viro's patchworking and an equivariant version of a deformation due to E.Horikawa.

## 1 Introduction

A *real algebraic variety* is a complex algebraic variety  $X$  equipped with an antiholomorphic involution  $c : X \rightarrow X$ . Such an antiholomorphic involution is called a *real structure* on  $X$ . The *real part* of  $(X, c)$ , denoted by  $\mathbb{R}X$ , is the set of points fixed by  $c$ . The *standard real structure*  $c_0$  on  $(\mathbb{C}^*)^n$  is defined by

$$c_0(Z_1, \dots, Z_n) = (\overline{Z_1}, \dots, \overline{Z_n}).$$

The *standard real structure* on a toric variety of dimension  $n$  is the real structure induced by the standard real structure on  $(\mathbb{C}^*)^n$ . In this text, the only real structures we consider on toric varieties are standard real structures. A *real subvariety* of a toric variety is a subvariety stable by the standard real structure. For example, a real algebraic surface in  $\mathbb{CP}^3$  is the zero set of a real homogeneous polynomial in 4 variables. Unless otherwise specified, all varieties considered are nonsingular. The homology is always considered with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients. For a topological space  $A$ , we put  $b_i(A) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(A, \mathbb{Z}/2\mathbb{Z})$ . The numbers  $b_i(A)$  are called *Betti numbers (with  $\mathbb{Z}/2\mathbb{Z}$  coefficients) of  $A$* . All polytopes considered are convex lattice polytopes in  $\mathbb{R}^n$ .

Let us remind several classical inequalities and congruences in topology of real algebraic varieties.

**Smith-Thom inequality and congruence:** Let  $X$  be a compact real algebraic variety. Then

$$b_*(\mathbb{R}X) \leq b_*(X) \text{ and } b_*(\mathbb{R}X) \equiv b_*(X) \pmod{2},$$

where  $b_*$  is the sum of all Betti numbers. The variety  $X$  is called an *M-variety* if  $b_*(\mathbb{R}X) = b_*(X)$  and an *(M - a)-variety* if  $b_*(\mathbb{R}X) = b_*(X) - 2a$ .

**Petrovsky-Oleinik inequalities:** Let  $X$  be a compact complex Kähler manifold of real dimension  $4n$  equipped with a real structure. Then

$$2 - h^{n,n}(X) \leq \chi(\mathbb{R}X) \leq h^{n,n}(X),$$

where  $\chi$  denotes the Euler characteristic and  $h^{p,q}$  denotes the  $(p, q)$ -Hodge number.

**Rokhlin congruence:** Let  $X$  be a compact  $M$ -variety of real dimension  $4n$ . Then

$$\chi(\mathbb{R}X) \equiv \sigma(X) \pmod{16},$$

where  $\sigma(X)$  is the signature of  $X$ .

**Gudkov-Kharlamov congruence:** Let  $X$  be a compact  $(M - 1)$ -variety of real dimension  $4n$ . Then

$$\chi(\mathbb{R}X) \equiv \sigma(X) \pm 2 \pmod{16}.$$

For an introduction concerning restrictions on the topology of real algebraic varieties, see [Wil78] or [DK00].

Let  $X$  be a compact connected simply-connected projective real surface. From the Smith-Thom inequality and the Petrovsky-Oleinik inequalities, one can deduce bounds for  $b_0(\mathbb{R}X)$  and  $b_1(\mathbb{R}X)$  in terms of Hodge numbers of  $X$ :

$$b_0(\mathbb{R}X) \leq \frac{1}{2}(h^{2,0}(X) + h^{1,1}(X) + 1), \quad (1)$$

$$b_1(\mathbb{R}X) \leq h^{2,0}(X) + h^{1,1}(X). \quad (2)$$

These bounds are not sharp in general. One can then ask the following questions.

**Question 1.** *What is the maximal possible value of  $b_0(\mathbb{R}X)$  for a real algebraic surface  $X$  in  $\mathbb{CP}^3$  of a given degree?*

**Question 2.** *What is the maximal possible value of  $b_1(\mathbb{R}X)$  for a real algebraic surface  $X$  in  $\mathbb{CP}^3$  of a given degree?*

If the degree is greater than 4, these questions are still widely open. In 1980, O.Viro formulated the following conjecture.

**Conjecture.** (*O.Viro*)

*Let  $X$  be a compact connected simply-connected projective real surface. Then*

$$b_1(\mathbb{R}X) \leq h^{1,1}(X).$$

This conjecture was an attempt to give an answer to Question 2. When  $X$  is the double covering of  $\mathbb{CP}^2$  ramified along a curve of an even degree, this conjecture is a reformulation of Ragsdale's conjecture (see [Vir80]). The first counterexample to Ragsdale's conjecture was constructed by I.Itenberg (see [Ite93]) using Viro's combinatorial patchworking (see Section 2 or [Ite97] or [Bih99]). This first counterexample opened the way to various counterexamples to Viro's conjecture and constructions of real algebraic surfaces with many connected components (see [Ite97], [Bih99], [Bih01], [Bru06], [IK96] and [Ore01]). It is not known whether Viro's conjecture is true for  $M$ -surfaces. In the case where  $X$  is a real algebraic surface of degree  $d$  in  $\mathbb{CP}^3$ , inequalities (1) and (2) specialize to the following ones:

$$b_0(\mathbb{R}X) \leq \frac{5}{12}d^3 - \frac{3}{2}d^2 + \frac{25}{12}d, \quad (1')$$

$$b_1(\mathbb{R}X) \leq \frac{5}{6}d^3 - 3d^2 + \frac{25}{6}d - 1. \quad (2')$$

Consider the case where  $X$  has degree 6. Then  $h^{1,1}(X) = 86$ , and the inequality (2'), combined with Petrovsky-Oleinik inequalities and Rokhlin congruence, gives  $b_1(\mathbb{R}X) \leq 94$ . F.Bihan constructed in [Bih99], using Viro's combinatorial patchworking, a real sextic  $X$  satisfying  $b_1(\mathbb{R}X) = 88$ . Moreover, the real part of  $X$  is homeomorphic to  $6S \amalg S_2 \amalg S_{42}$ , where  $S$  denotes a 2-dimensional sphere and  $aS_\alpha$  denotes the disjoint union of  $a$  spheres each having  $\alpha$  handles. In this note, we improve this construction.

**Theorem 1.** *There exists a real sextic surface  $X$  in  $\mathbb{CP}^3$  satisfying  $b_1(\mathbb{R}X) = 90$  such that*

$$\mathbb{R}X \simeq 4S \amalg 2S_2 \amalg S_{41}.$$

The paper is organized as follows. In Sections 2 and 3, we remind Viro's method and some results about the Euler characteristic of T-surfaces. In Section 4, we describe a class of real algebraic surfaces, the so-called surfaces of type (1c), and an equivariant deformation of a real surface of type (1c) to a real sextic surface. In Section 5, we use Viro's combinatorial patchworking to construct a real surface  $Z$  of type (1c). Then, using the general Viro's method, we slightly modify the construction of  $Z$  to obtain a real surface  $Y$  of type (1c) satisfying

$$\mathbb{R}Y \simeq 4S \amalg 2S_2 \amalg S_{41}.$$

The existence of a real sextic surface  $X$  satisfying  $92 \leq b_1(\mathbb{R}X) \leq 94$  is still unknown.

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## 2 Viro's method

### 2.1 T-construction

The combinatorial patchworking construction (or T-construction) works in any dimension.

Let  $(u_1, \dots, u_n)$  be coordinates in  $\mathbb{R}^n$ , and let  $\Delta$  be a  $n$ -dimensional polytope in  $\mathbb{R}_+^n$ , where  $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ . Denote by  $Tor(\Delta)$  the toric variety associated with  $\Delta$ . We denote by  $\mathbb{R}Tor(\Delta)$  the real part of  $Tor(\Delta)$  for the standard real structure. Take a triangulation  $\tau$  of  $\Delta$  with vertices having integer coordinates, and a distribution of signs at the vertices of  $\tau$ . Denote the sign at any vertex  $(i_1, \dots, i_n)$  by  $\delta_{i_1, \dots, i_n}$ . For  $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ , let  $s_\epsilon$  be the symmetry of  $\mathbb{R}^n$  defined by

$$s_\epsilon(u_1, \dots, u_n) = ((-1)^{\epsilon_1}u_1, \dots, (-1)^{\epsilon_n}u_n).$$

Denote by  $\Delta_*$  the union

$$\cup_{\epsilon \in (\mathbb{Z}/2\mathbb{Z})^n} s_\epsilon(\Delta).$$

Extend the triangulation  $\tau$  to a symmetric triangulation of  $\Delta_*$ , and the distribution of signs  $\delta_{i_1, \dots, i_n}$  to a distribution at the vertices of the extended triangulation using the following formula:

$$\delta_{s_\epsilon(i_1, \dots, i_n)} = \left( \prod_{j=1}^{j=n} (-1)^{\epsilon_j i_j} \right) \delta_{i_1, \dots, i_n}.$$

If a tetrahedron  $T$  of the triangulation of  $\Delta_*$  has vertices of different signs, denote by  $S_T$  the convex hull of the middle points of the edges of  $T$  having endpoints of opposite signs. Denote by  $S$  the union of all such  $S_T$ . It is a  $(n-1)$  piecewise-linear manifold contained in  $\Delta_*$ . If  $\Gamma$  is a face of  $\Delta_*$ , then, for all integer vectors  $\alpha$  orthogonal to  $\Gamma$ , identify  $\Gamma$  with  $s_\alpha(\Gamma)$ . Denote by  $\hat{\Delta}$  the quotient of  $\Delta_*$  under these identifications, and by  $\pi_\Delta$  the quotient map. The real part  $\mathbb{R} \text{Tor}(\Delta)$  is homeomorphic to  $\hat{\Delta}$ .

The triangulation  $\tau$  of  $\Delta$  is said to be *convex* if there exists a convex piecewise-linear function  $\nu : \Delta \rightarrow \mathbb{R}$  whose domains of linearity coincide with the tetrahedra of  $\tau$ .

**Theorem 2.** (*O. Viro*)

*Assume that the only singularities of  $\text{Tor}(\Delta)$  correspond to the vertices of  $\Delta$  and that the triangulation  $\tau$  of  $\Delta$  is convex. Then there exists a nonsingular real algebraic hypersurface  $X$  in  $\text{Tor}(\Delta)$  belonging to the linear system associated with  $\Delta$ , and a homeomorphism  $\mathbb{R} \text{Tor}(\Delta) \rightarrow \hat{\Delta}$  mapping  $\mathbb{R}X$  to  $\pi_\Delta(S)$ .*

A polynomial defining such an hypersurface  $X$  can be written down explicitly. If  $t > 0$  is sufficiently small, the polynomial

$$\sum_{(i_1, \dots, i_n) \in V} \delta_{i_1, \dots, i_n} \prod_{j=1}^{j=n} \left( x_j^{i_j} \right) t^{\nu(i_1, \dots, i_n)} \quad (3)$$

(where  $V$  is the set of vertices of  $\tau$  and  $\nu$  is a function ensuring the convexity of  $\tau$ ) defines an hypersurface in  $(\mathbb{C}^*)^n$ , such that the compactification of this hypersurface in  $\text{Tor}(\Delta)$  has the properties described in Theorem 2.

**Definition 1.** *A polynomial of the form (3) is called a Viro polynomial and an hypersurface defined by such a polynomial (for sufficiently small  $t > 0$ ) is called a T-hypersurface.*

**Remark 1.** *The assumption on the singularities of  $\text{Tor}(\Delta)$  is not essential. See Section 2.2.*

The T-construction is a particular case of a more general construction, called Viro's patchworking or Viro's method.

## 2.2 General Viro's method

In this construction, we glue together more complicated pieces than before. These pieces are called *charts of polynomials*.

**Definition 2.** Let  $f$  be a polynomial in  $\mathbb{R}[x_1, \dots, x_n]$  and  $Z(f)$  be the set  $\{x \in (\mathbb{R}^*)^n \mid f(x) = 0\}$ . Let  $\Delta(f) \subset (\mathbb{R}_+)^n$  be the Newton polygon of  $f$ . In the octant  $(\mathbb{R}_+^*)^n$ , we define  $\phi$  as

$$\begin{aligned} \phi : (\mathbb{R}_+^*)^n &\rightarrow (\mathbb{R}^*)^n \\ z &\mapsto \frac{\sum_{i \in \Delta \cap \mathbb{Z}^n} |z^i| i}{\sum_{i \in \Delta \cap \mathbb{Z}^n} |z^i|}. \end{aligned}$$

In the octant  $s_\epsilon((\mathbb{R}_+^*)^n)$ , we put

$$\phi(s_\epsilon(z)) = s_\epsilon(\phi(z)),$$

where  $s_\epsilon(x_1, \dots, x_n) = ((-1)^{\epsilon_1} x_1, \dots, (-1)^{\epsilon_n} x_n)$ .

We call chart of  $f$  the closure of  $\phi(Z(f))$  in  $\Delta(f)_*$ . Denote by  $C(f)$  the chart of  $f$ .

**Definition 3.** Let  $f = \sum a_i x^i$  be a polynomial in  $n$  variables. Let  $\Gamma \subset \mathbb{Z}^n$  be a subset of the Newton polygon  $\Delta(f)$  of  $f$ . The truncation of  $f$  to  $\Gamma$  is the polynomial  $f^\Gamma$  defined by  $f^\Gamma = \sum_{i \in \Gamma} a_i x^i$ .

**Definition 4.** A polynomial  $f$  is called non-degenerated with respect to its Newton polygon  $\Delta(f)$  if for any face  $\Gamma$  of  $\Delta(f)$  (including  $\Delta(f)$  itself), the polynomial  $f^\Gamma$  defines a nonsingular hypersurface in  $(\mathbb{C}^*)^k$ , where  $k$  is the dimension of  $\Gamma$ .

Let  $\Delta$  be an  $n$ -dimensional polytope in  $\mathbb{R}_+^n$  and let  $\cup_{i \in I} \Delta_i$  be a decomposition of  $\Delta$  such that all the  $\Delta_i$  have vertices with integer coordinates. For any  $i \in I$ , take a polynomial  $f_i$  such that the  $f_i$ 's verify the following properties:

- for all  $i \in I$ , the Newton polygon of  $f_i$  is  $\Delta_i$ ,
- if  $\Gamma = \Delta_i \cap \Delta_j$ , then  $f_i^\Gamma = f_j^\Gamma$ ,
- for all  $i \in I$ , the polynomial  $f_i$  is non-degenerated with respect to  $\Delta_i$ .

The polynomials  $f_i$  define a unique polynomial  $f = \sum_{w \in \Delta \cap \mathbb{Z}^n} a_w x^w$ , such that  $f^{\Delta_i} = f_i$  for all  $i \in I$ . The decomposition  $\cup_{i \in I} \Delta_i$  of  $\Delta$  is said to be *convex* if there exists a convex piecewise-linear function  $\nu : \Delta \rightarrow \mathbb{R}$  whose domains of linearity coincide with the  $\Delta_i$ .

**Theorem 3.** (*O. Viro*)

Assume that the decomposition  $\cup_{i \in I} \Delta_i$  of  $\Delta$  is convex and let  $\nu : \Delta \rightarrow \mathbb{R}$  be a function certifying its convexity. Define the associated Viro polynomial  $f_t = \sum_{w \in \Delta \cap \mathbb{Z}^n} a_w t^{\nu(w)} x^w$ . Then there exists  $t_0 > 0$  such that if  $0 < t < t_0$ , then  $f_t$  is non-degenerated with respect to  $\Delta$  and there exists an homeomorphism of  $\hat{\Delta}$  sending  $\pi_{\Delta(f)}(C(f_t))$  to  $\pi_{\Delta(f)}(\cup_{i \in I} C(f_i))$ .

For more details about the general Viro's method, see for example [Vir84] or [Ris93].

### 3 Euler characteristic of the real part of a T-surface

We remind in this section some results about the topology of T-surfaces. Let us introduce first some terminology concerning simplices and triangulations of polytopes.

**Definition 5.** *The integer volume of an  $n$ -dimensional simplex in  $\mathbb{R}^n$  is equal to  $n!$  times its euclidean volume. An  $n$ -dimensional simplex in  $\mathbb{R}^n$  is called maximal if it does not contain other integer points than its vertices. A maximal simplex is called primitive if its integer volume is equal to 1 and elementary if its integer volume is odd.*

**Definition 6.** *A triangulation of an  $n$ -dimensional polytope  $P$  in  $\mathbb{R}^n$  is called maximal (resp., primitive) if all  $n$ -dimensional simplices in the triangulation are maximal (resp., primitive).*

**Definition 7.** *The star of a face  $F$  in a triangulation  $\tau$ , denoted by  $st(F)$ , is the union of all simplices in  $\tau$  having  $F$  as face.*

**Definition 8.** *We say that an edge  $\lambda$  of a triangulation  $\tau$  is of length  $n$  if  $\lambda$  contains  $n + 1$  integer points.*

**Definition 9.** *Let  $\tau$  be a triangulation containing an edge  $\lambda$  of length 2. Suppose that  $\lambda$  is the only edge of length greater than 1 in  $st(\lambda)$ . The refined triangulation is obtained by adding the middle point of  $\lambda$  to the set of vertices of  $\tau$  and by subdividing each tetrahedron in  $st(\lambda)$  accordingly.*

Let  $\Delta$  be a 3-dimensional polytope in  $\mathbb{R}_+^3$ . Suppose that the only singularities of  $Tor(\Delta)$  correspond to the vertices of  $\Delta$ . The real part of a T-surface in  $Tor(\Delta)$  admits a cellular decomposition coming from the triangulation of  $\hat{\Delta}$ . This cellular decomposition allows one to compute the Euler characteristic of the real part.

**Proposition 1.** (see [Bih99])

*Suppose that  $\Delta$  admits a maximal triangulation  $\tau$ . Given a distribution of signs  $D(\tau)$ , denote by  $N$  (resp.,  $P$ ) the set of tetrahedra of even volume in  $\tau$  with negative (resp., positive) product of signs at the vertices. Let  $E$  be the set of elementary tetrahedra in  $\tau$ . Let  $Z$  be a T-surface obtained from  $(\tau, D(\tau))$ . Then*

$$\chi(\mathbb{R}Z) = \sigma(\mathbb{C}Z) + \sum_{T \text{ tetrahedra in } \tau} (Vol(T) - \varepsilon_T),$$

*where  $\varepsilon_T = 0, 1, 2$  if  $T \in N, E, P$  respectively.*

**Proposition 2.** (see [Bih99])

*Suppose that  $\Delta$  admits a triangulation  $\tau$  with an edge  $\lambda$  of length 2 (with middle point  $a$ ) such that  $\lambda$  is the only edge of length greater than 1 in  $st(\lambda)$ . Denote by  $k$  the dimension of the minimal face of  $\Delta$  containing  $\lambda$ . Denote by  $\tau_a$  the refined triangulation (see Definition 9). Let  $D(\tau)$  be any distribution of signs in  $\tau$  and extend it to  $D(\tau_a)$  choosing any sign of  $a$ . Let  $P_a$  be the set of tetrahedra in  $st(a)$  which are of even volume and positive product of signs at the vertices. Let  $E_a$  be the set of elementary tetrahedra in  $st(a)$ . Denote by  $Z$ , resp.  $Z_a$ , a*

$T$ -surface obtain from  $(\tau, D(\tau))$ , resp.  $(\tau_a, D(\tau_a))$ .

If the endpoints of  $\lambda$  have opposite signs, then  $\chi(\mathbb{R}Z) = \chi(\mathbb{R}Z_a)$ , and

$$\chi(\mathbb{R}Z) - \chi(\mathbb{R}Z_a) = \#(E_a) + 2\#(P_a) - 2^k,$$

otherwise.

## 4 An equivariant deformation

In his construction, Bihan used an equivariant version of Horikawa's deformation of surfaces of type (1c) in  $\mathbb{CP}^4(2)$  (see [Hor93]).

**Definition 10.** A *family of compact complex surfaces*  $\mathcal{F} = (L, p, B)$  consists of a pair of connected complex manifolds  $L$  and  $B$ , and a proper holomorphic map  $p : L \rightarrow B$  which is a submersion and whose fibers  $L_b$  are connected surfaces.

Let  $V$  be a connected compact complex surface. An **elementary deformation** of  $V$  parametrised by a complex contractible manifold  $B$  consists of a connected complex manifold  $L$ , a base point  $b_0 \in B$ , a family  $\mathcal{F} = (L, p, B)$  and an injective morphism  $i : V \rightarrow L$  such that  $i(V) = L_{b_0}$ .

A **result of an elementary deformation of  $V$**  is a connected complex surface which is a fiber of the map  $p$ .

On the set of complex surfaces, introduce the equivalence relation generated by elementary deformations and isomorphisms. Any surface belonging to the equivalent class of  $V$  is called a **deformation** of  $V$ .

Suppose that  $(V, c)$  is a real surface. An **elementary equivariant deformation** of  $(V, c)$  is an elementary deformation of  $V$  such that  $L$  (resp.,  $B$ ) is equipped with an antiholomorphic involution  $\text{Conj} : L \rightarrow L$  (resp.,  $\text{conj} : B \rightarrow B$ ) satisfying  $p \circ \text{Conj} = \text{conj} \circ p$ ,  $\text{conj}(b_0) = b_0$  and  $\text{Conj} \circ i = i \circ c$ .

On the set of real surfaces, introduce the equivalence relation generated by elementary equivariant deformations and real isomorphisms.

Consider the 4-dimensional weighted projective space  $\mathbb{CP}^4(2)$  with complex homogeneous coordinates  $Z_0, Z_1, Z_2, Z_3$  of weight 1 and  $Z_4$  of weight 2.

**Definition 11.** (see [Hor93])

An algebraic surface  $Y$  in  $\mathbb{CP}^4(2)$  is said to be of type (1c) if  $Y$  is defined by the following system of equations:

$$\begin{cases} Z_4^3 + f_2(Z)Z_4^2 + f_4(Z)Z_4 + f_6(Z) = 0, \\ Z_0Z_3 - Z_1Z_2 = 0. \end{cases}$$

where  $f_{2i}(Z)$  is a homogeneous polynomial of degree  $2i$  in the variables  $Z_0, Z_1, Z_2, Z_3$ .

We define a real algebraic surface of type (1c) to be a complex algebraic surface of type (1c) invariant under the standard real structure on  $\mathbb{CP}^4(2)$ .

In [Hor93], Horikawa showed that any nonsingular algebraic surface of type (1c) can be deformed to a nonsingular surface of degree 6 in  $\mathbb{CP}^3$ . The same result is true in the real category.

**Proposition 3.** (see [Bih01])

Let  $Y$  be a nonsingular real algebraic surface of type (1c). Then, there exists an equivariant deformation of  $Y$  to a nonsingular real surface  $X$  of degree 6 in  $\mathbb{CP}^3$ .

*Proof.* Consider the elementary equivariant deformation of  $Y = Y_0$  determined by the family  $(Y_\epsilon)$  for  $\epsilon \in \mathbb{R}$ , where  $Y_\epsilon$  is defined by the following system of equations:

$$\begin{cases} Z_4^3 + f_2(Z)Z_4^2 + f_4(Z)Z_4 + f_6(Z) = 0, \\ Z_0Z_3 - Z_1Z_2 - \epsilon Z_4 = 0. \end{cases}$$

As  $Y$  is a nonsingular surface, then for sufficiently small  $\epsilon$ , the surface  $Y_\epsilon$  is nonsingular. The system defining the surface  $Y_\epsilon$  can be transformed into:

$$\begin{cases} (\frac{Z_0Z_3 - Z_1Z_2}{\epsilon})^3 + f_2(Z)(\frac{Z_0Z_3 - Z_1Z_2}{\epsilon})^2 + f_4(Z)(\frac{Z_0Z_3 - Z_1Z_2}{\epsilon}) + f_6(Z) = 0, \\ Z_4 = \frac{Z_0Z_3 - Z_1Z_2}{\epsilon}. \end{cases}$$

Now, consider the projection

$$\begin{aligned} p : \mathbb{CP}^4(2) \setminus \{(0 : 0 : 0 : 0 : 1)\} &\rightarrow \mathbb{CP}^3 \\ (Z_0 : Z_1 : Z_2 : Z_3 : Z_4) &\mapsto (Z_0 : Z_1 : Z_2 : Z_3). \end{aligned}$$

The point  $(0 : 0 : 0 : 0 : 1) \in \mathbb{CP}^4(2)$  does not belong to  $Y_\epsilon$ , hence  $p|_{Y_\epsilon}$  is well defined. The projection  $p$  produces a complex isomorphism between  $Y_\epsilon$  and the algebraic surface  $X_\epsilon$  of degree 6 in  $\mathbb{CP}^3$  defined by the polynomial

$$(\frac{Z_0Z_3 - Z_1Z_2}{\epsilon})^3 + f_2(Z)(\frac{Z_0Z_3 - Z_1Z_2}{\epsilon})^2 + f_4(Z)(\frac{Z_0Z_3 - Z_1Z_2}{\epsilon}) + f_6(Z) = 0.$$

Moreover, this isomorphism is equivariant with respect to the involution  $c$  and the standard involution on  $\mathbb{CP}^3$ .  $\square$

**Remark 2.** This deformation can be geometrically understood as a deformation of  $\mathbb{CP}^3$  to the normal cone of a nonsingular quadric. (See [Ful98] for the general process of deforming an algebraic variety to the normal cone of a subvariety).

**Remark 3.** Any surface of type (1c) is a hypersurface in the quadric defined by the equation  $(Z_0Z_3 - Z_1Z_2 = 0)$  in  $\mathbb{CP}^4(2)$ . This quadric is a projective toric variety. In particular, one may use Viro's patchworking to produce real surfaces in  $Q$ . A natural polytope which may be used to apply Viro's patchworking to produce real algebraic surfaces of type (1c) is the polytope  $Q$  with vertices  $(0, 0, 0), (6, 0, 0), (6, 6, 0), (0, 6, 0), (0, 0, 3)$  in  $\mathbb{R}^3$  (see Figure 1).

## 5 Construction of a surface $X$ of degree 6 with 45 handles

**Proposition 4.** There exists a real algebraic surface  $Y$  of type (1c) such that

$$\mathbb{R}Y \simeq 4S \amalg 2S_2 \amalg S_{41}.$$



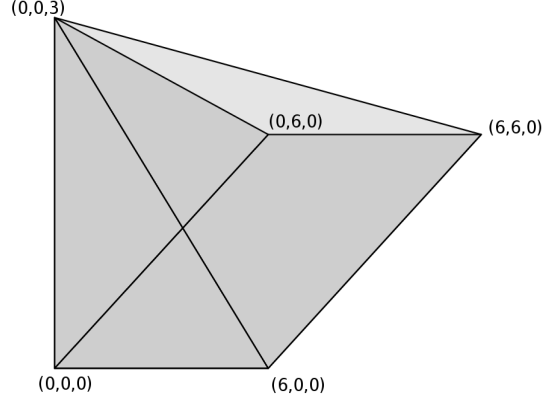


Figure 1: Polytope  $Q$ .

*Proof of Theorem 1.* Performing the equivariant deformation described in Proposition 3 to the surface  $Y$ , we obtain a real sextic surface  $X$  in  $\mathbb{CP}^3$ , such that

$$\mathbb{R}X \simeq 4S \amalg 2S_2 \amalg S_{41}.$$

□

The rest of the article is devoted to the proof of Proposition 4. Our strategy is first to describe a T-construction of an auxilliary surface  $Z$  of Newton polytope  $Q$ . Then, we use the general Viro's patchworking method to modify slightly the construction.

### 5.1 The auxilliary surface $Z$

We describe a triangulation  $\tau$  of  $Q$  and a distribution of signs  $D(\tau)$  at the vertices of  $\tau$ . Consider the cone  $C$  with vertex  $(1, 0, 2)$  over the square  $Q_0 = Q \cap \{w = 0\}$  (see Figure 2). Take any primitive convex triangulation of  $Q_0$  containing the edges depicted in Figure 3. Then, triangulate  $C$  into the cones with vertex  $(1, 0, 2)$  over the triangles of the triangulation of  $Q_0$ . The triangulation of the cone  $C$  contains 12 edges of length 2 (edges joining  $(1, 0, 2)$  to the points of coordinates  $(1, 0) \bmod 2$  inside  $Q_0$ ). For the three edges  $[(1, 0, 2) - (1, 0, 0)]$ ,  $[(1, 0, 2) - (3, 0, 0)]$  and  $[(1, 0, 2) - (5, 0, 0)]$  of length 2, refine the triangulation as explained in Definition 9.

Consider the tetrahedra  $\alpha_1$  and  $\alpha_2$  with vertices  $(1, 0, 2), (6, 6, 0), (4, 0, 1), (6, 0, 0)$  and  $(1, 0, 2), (0, 6, 0), (0, 0, 1), (0, 0, 0)$  respectively. See Figure 4 for a picture of  $\alpha_1$ . Triangulate  $\alpha_1$  into the cones with vertex  $(4, 0, 1)$  over the triangles in the triangulation of the triangle with vertices  $(1, 0, 2), (6, 6, 0), (6, 0, 0)$ . Triangulate  $\alpha_2$  into the cones with vertex  $(0, 0, 1)$  over the triangles in the triangulation

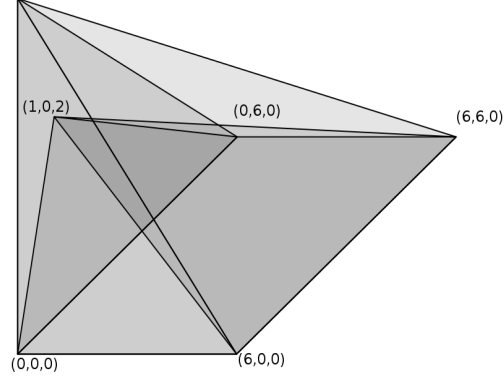


Figure 2: Cone  $C$ .

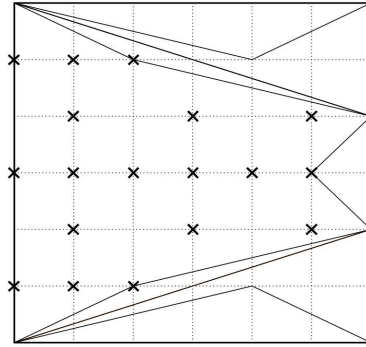


Figure 3: The fixed part of a triangulation of  $Q_0$  and the distribution of signs. A point gets a sign  $+$  if and only if it is ticked.

of the triangle with vertices  $(1, 0, 2)$ ,  $(0, 6, 0)$ ,  $(0, 0, 0)$ . All the tetrahedra of the triangulations constructed are primitive.

Consider the tetrahedra  $\beta_1$  and  $\beta_2$  with vertices  $(1, 0, 2)$ ,  $(6, 6, 0)$ ,  $(4, 4, 1)$ ,  $(4, 0, 1)$  and  $(1, 0, 2)$ ,  $(0, 6, 0)$ ,  $(0, 4, 1)$ ,  $(0, 0, 1)$  respectively. See Figure 5 for a picture of  $\beta_1$ . Triangulate  $\beta_1$  and  $\beta_2$  into 4 tetrahedra, respectively, using the subdivision of the segment  $[(4, 4, 1) - (4, 0, 1)]$  and  $[(0, 4, 1) - (0, 0, 1)]$  into four primitive edges. All the tetrahedra of the triangulations of  $\beta_1$  and  $\beta_2$  are primitive.

Consider the tetrahedron  $\gamma_1$  with vertices  $(1, 0, 2)$ ,  $(6, 6, 0)$ ,  $(4, 4, 1)$ ,  $(0, 4, 1)$ , see Figure 6. Triangulate  $\gamma_1$  into 4 tetrahedra, using the subdivision of the segment  $[(4, 4, 1) - (0, 4, 1)]$ . All the tetrahedra of the triangulation of  $\gamma_1$  are of volume 2.

Consider the tetrahedron  $\gamma_2$  with vertices  $(1, 0, 2)$ ,  $(6, 6, 0)$ ,  $(0, 6, 0)$ ,  $(0, 4, 1)$ . The triangle with vertices  $(1, 0, 2)$ ,  $(6, 6, 0)$ ,  $(0, 6, 0)$  is already triangulated. Use this triangulation to subdivide  $\gamma_2$ . Finally, for the three edges  $[(1, 0, 2) - (1, 6, 0)]$ ,

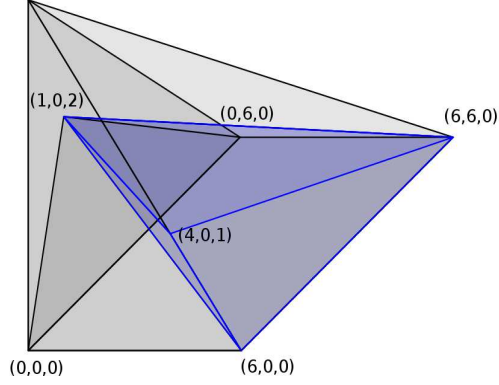


Figure 4: Tetrahedron  $\alpha_1$ .

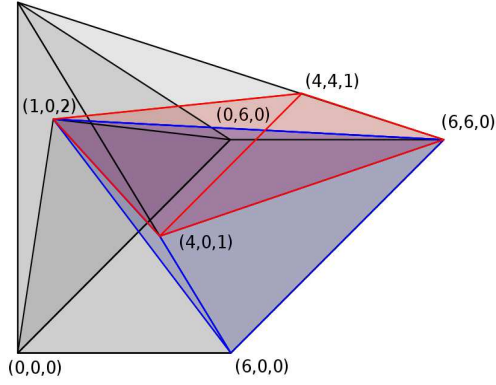


Figure 5: Tetrahedron  $\beta_1$ .

$[(1, 0, 2) - (3, 6, 0)]$  and  $[(1, 0, 2) - (5, 6, 0)]$  of length 2, refine the triangulation as explained in Definition 9.

At the present time, the part lying under the cone with vertex  $(1, 0, 2)$  over  $Q \cap \{w = 1\}$  is triangulated (see Figure 7). Consider the pentagon  $P$  with vertices  $(1, 0, 2)$ ,  $(2, 0, 2)$ ,  $(2, 2, 2)$ ,  $(1, 2, 2)$ ,  $(0, 1, 2)$ , triangulate it with any primitive convex triangulation and consider the two cones over it with vertex  $(0, 0, 3)$  and  $(4, 4, 1)$  respectively (see Figure 8). Complete the triangulation considering the following tetrahedra:

- The joint of the segment  $[(4, 0, 1) - (4, 4, 1)]$  and  $[(1, 0, 2) - (2, 0, 2)]$  trian-

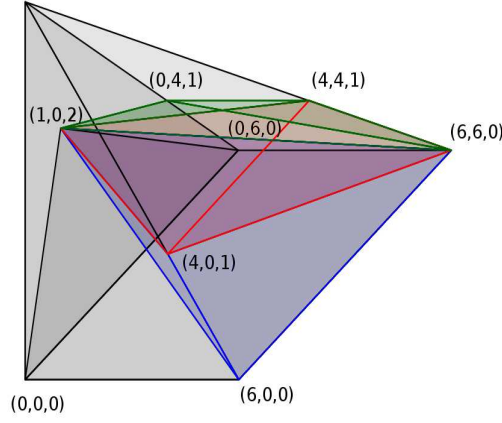


Figure 6: Tetrahedron  $\gamma_1$ .

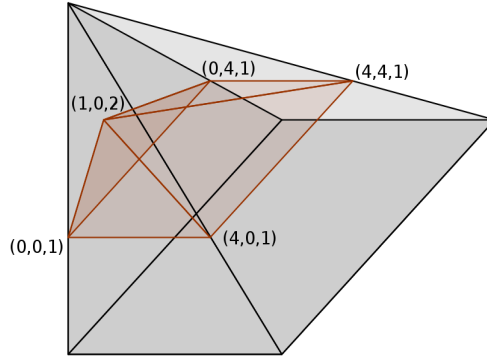


Figure 7: Cone over  $Q \cap \{w = 1\}$ .

gulated into 4 primitive tetrahedra, using the triangulation of the segment  $[(4, 0, 1) - (4, 4, 1)]$  into 4 edges.

- The joint of the segment  $[(0, 4, 1) - (4, 4, 1)]$  and  $[(0, 1, 2) - (0, 2, 2)]$  triangulated into 4 primitive tetrahedra, using the triangulation of the segment  $[(0, 4, 1) - (4, 4, 1)]$  into 4 edges.
- The joint of the segment  $[(0, 4, 1) - (4, 4, 1)]$  and  $[(1, 0, 2) - (0, 1, 2)]$  triangulated into 4 primitive tetrahedra, using the triangulation of the segment  $[(0, 4, 1) - (4, 4, 1)]$  into 4 edges.

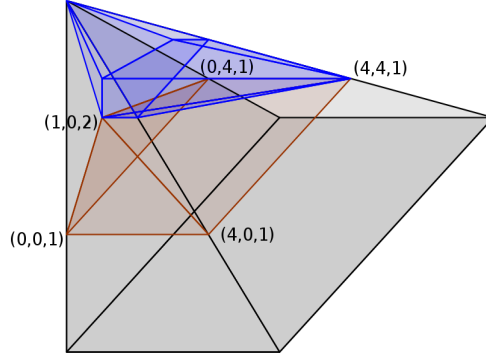


Figure 8: Cones over the pentagon  $P$ .

- The two cones over the triangle  $(0,0,2), (1,0,2), (0,1,2)$  with vertices  $(0,0,1)$  and  $(0,0,3)$ , respectively.
- The two cones over the triangle  $(0,1,2), (0,2,2), (1,2,2)$  with vertices  $(0,4,1)$  and  $(0,0,3)$ , respectively.

Denote by  $\rho$  the obtained subdivision of  $Q$ . To show the convexity of  $\rho$ , one can proceed as in [Ite97]. First, remark that the “coarse” subdivision given by the cone  $C$ , the tetrahedra  $\alpha_i$ , the tetrahedra  $\beta_i$ , the tetrahedra  $\gamma_i$ , the cones over the pentagon  $S$  and the remaining three joints and two cones is convex. Denote by  $\nu'$  a convex piecewise-linear function certifying the convexity of this “coarse” subdivision.

Choose three convex functions  $\nu_1, \nu_2$  and  $\nu_3$  certifying the convexity of the subdivision of the three edges  $[(0,0,1) - (0,4,1)]$ ,  $[(0,4,1) - (4,4,1)]$  and  $[(4,4,1) - (4,0,1)]$ . Choose also a convex function  $\nu_4$  certifying the convexity of the chosen subdivision of the pentagon and a convex function  $\nu_5$  certifying the convexity of the chosen subdivision of the cone  $C$ .

Consider a piecewise-linear function  $\nu : Q \rightarrow \mathbb{R}$  which is affine-linear on each tetrahedron of the subdivision  $\rho$  and takes the value  $\nu'(x) + \sum \epsilon_i \nu_i(x)$  at every vertex  $x$ . The function  $\nu$  for positive sufficiently small  $\epsilon_i$  certifies the convexity of the subdivision  $\rho$ .

Define the distribution of signs  $D(\tau)$  at the vertices of  $\tau$ . For the points inside  $Q_0$ , take the distribution of signs shown in Figure 3. Denote by  $A$  a T-curve in  $\mathbb{P}^1 \times \mathbb{P}^1$  obtained from the triangulation  $\tau$  and the distribution  $D(\tau)$  restricted to  $Q_0$ . The chart of  $A$  is depicted in Figure 12 b). The distribution of signs at the vertices of  $\tau$  belonging to  $Q \cap \{w \geq 1\}$  is summarized in Figure 9. The point  $(0,0,3)$  gets the sign  $+$ .

Let us compute the Euler characteristic  $\chi(\mathbb{R}Z)$  of  $\mathbb{R}Z$ . The triangulation  $\tau$  contains 6 edges of length 2 with endpoints of opposite signs, and some tetrahedra of volume 2 in  $\gamma_1$  and in the cone  $C$ . Since all the other tetrahedra are

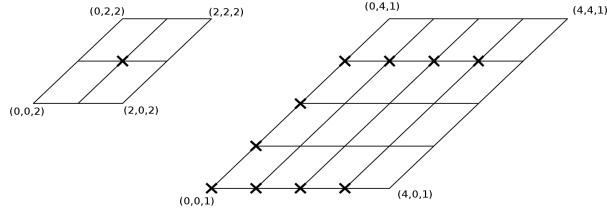


Figure 9: Distribution of signs for  $w = 1$  and  $w = 2$ . A point gets a sign  $+$  if and only if it is ticked.

elementary and the stars of the four edges of length 2 are disjoint, we can use Propositions 1 and 2 to compute  $\chi(\mathbb{R}Z)$ . In  $\gamma_1$  all the signs are positive, and in the cone  $C$ , six tetrahedra of volume 2 have negative product of signs. One obtains:

$$\chi(\mathbb{R}Z) = \sigma(\mathbb{C}Z) + 12 = -52.$$

## 5.2 The surface $Y$

To construct the surface  $Y$ , we use a real trigonal curve ( $C_3 = 0$ ) constructed by E. Brugallé in [Bru06]. The Newton polygon of the polynomial  $C_3$  is  $\text{Conv}((0, 0), (6, 0), (0, 3), (6, 1))$  and the chart of  $C_3$  is depicted in Figure 10.

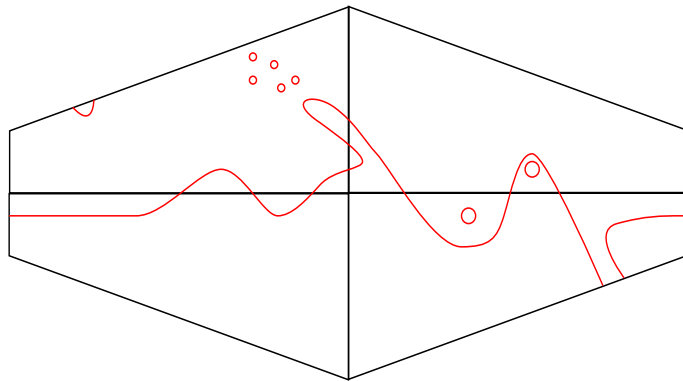


Figure 10: Chart of  $(C_3 = 0)$ .

Denote by  $\Gamma$  the hexagon  $\text{Conv}((0, 0), (4, 1), (6, 2), (6, 4), (4, 5), (0, 6))$ . Consider the charts of the polynomials

- $Y^3 C_3(X, Y)$ ,  $Y^3 C_3(X, \frac{1}{Y})$ ,

- $Y^6 b(X^3 \frac{1}{Y}, X^4 \frac{1}{Y}), b(X^3 Y, X^4 Y),$

where  $b(X, Y) = Y + (X + x_1)(X + x_2)$ , with  $x_1, x_2 > 0$  appropriately chosen so that the restrictions of the polynomials  $C_3(X, Y)$  and  $Y^3 b(X^3 \frac{1}{Y}, X^4 \frac{1}{Y})$  to  $\text{Conv}((0, 3); (6, 1))$  are equal. By Viro's patchworking theorem, there exists a

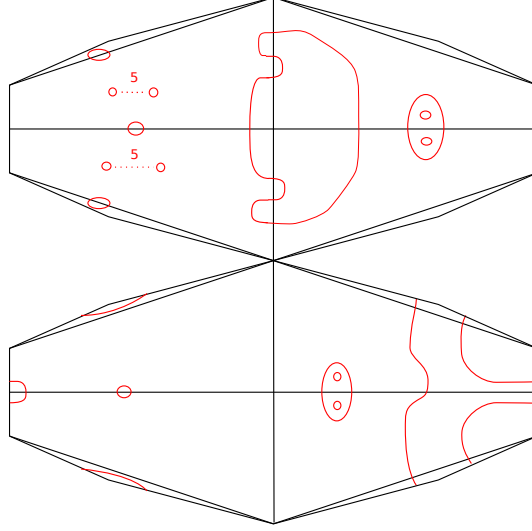


Figure 11: Chart of the polynomial  $P$ .

polynomial  $P$  of Newton polygon  $\Gamma$  whose chart is depicted in Figure 11. To construct the surface  $Y$ , apply the general Viro's patchworking inside  $Q$  with

- the chart of  $xz^2 + P(x, y)$  inside  $\text{Conv}(\Gamma, (1, 0, 2))$ ,
- the same triangulation and distribution of signs as in Section 5.1 outside  $\text{Conv}(\Gamma, (1, 0, 2))$ .

Denote by  $\hat{A}$  the curve in  $\mathbb{P}^1 \times \mathbb{P}^1$  obtained as the intersection of  $Y$  with the toric divisor corresponding to the face  $Q_0$ . See Figure 12 a).

Let us now compute the Euler characteristic of  $\mathbb{R}Y$ . To compute it, we compare the Euler characteristics of  $\mathbb{R}Z$  and  $\mathbb{R}Y$ . First of all, denote  $Z_1$  (resp.,  $Y_1$ ) the surfaces constructed in the same way as  $Z$  (resp.,  $Y$ ) but where the six edges  $[(1, 0, 2) - (1, 0, 0)]$ ,  $[(1, 0, 2) - (3, 0, 0)]$ ,  $[(1, 0, 2) - (5, 0, 0)]$ ,  $[(1, 0, 2) - (1, 6, 0)]$ ,  $[(1, 0, 2) - (3, 6, 0)]$  and  $[(1, 0, 2) - (5, 6, 0)]$  are not refined. From Proposition 2, one obtains

$$\chi(\mathbb{R}Y) - \chi(\mathbb{R}Z) = \chi(\mathbb{R}Y_1) - \chi(\mathbb{R}Z_1).$$

Then, notice that outside of  $C$ , the triangulation and distribution of signs defining  $Z_1$  and  $Y_1$  coincide. Denote by  $Z_2$  (resp.,  $Y_2$ ) the surfaces with Newton polygon  $C$ , defined by  $(A(x, y) + xz^2 = 0)$  (resp.,  $(\hat{A}(x, y) + xz^2 = 0)$ ) and compactified in  $\text{Tor}(C)$ . These surfaces are singular, with 12 ordinary double points. However, there exist two homeomorphic compact sets  $B \subset \mathbb{R}\text{Tor}(Q)$  and  $B' \subset \mathbb{R}\text{Tor}(C)$  such that:

- $\mathbb{R}Y_1 \setminus B$  is homeomorphic to  $\mathbb{R}Z_1 \setminus B$ ,
- $\mathbb{R}Y_2 \setminus B'$  is homeomorphic to  $\mathbb{R}Z_2 \setminus B'$ ,

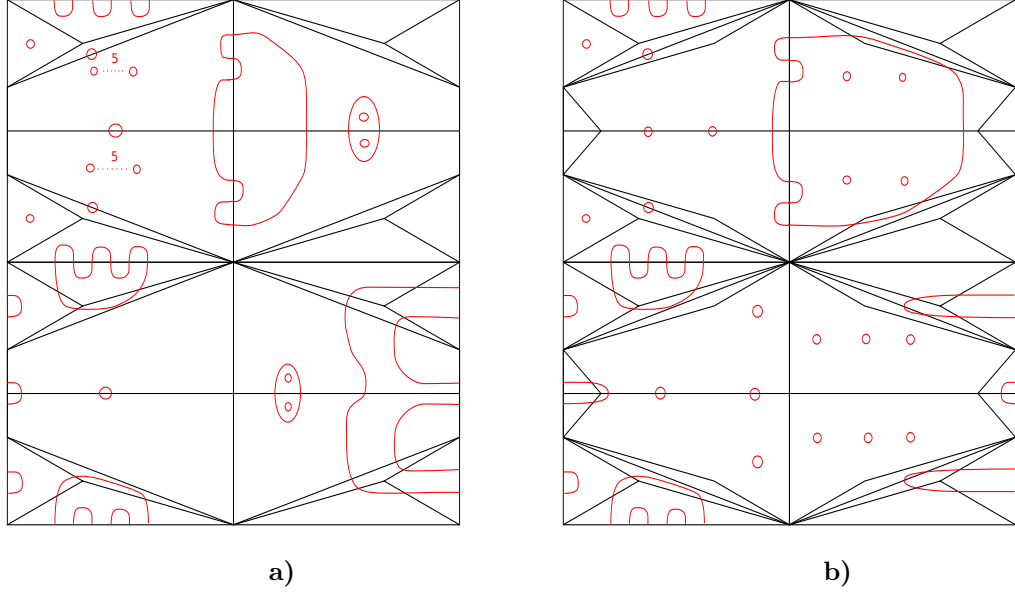


Figure 12: a):  $\mathbb{R}\hat{A}$     b):  $\mathbb{R}A$

- $\mathbb{R}Y_1 \cap B$  is homeomorphic to  $\mathbb{R}Y_2 \cap B'$ ,
- $\mathbb{R}Z_1 \cap B$  is homeomorphic to  $\mathbb{R}Z_2 \cap B'$ .

So one has:

$$\chi(\mathbb{R}Y_2 \cap B') - \chi(\mathbb{R}Z_2 \cap B') = \chi(\mathbb{R}Y_1 \cap B) - \chi(\mathbb{R}Z_1 \cap B).$$

By the additivity of the Euler characteristic, one also has that

$$\chi(\mathbb{R}Y_1 \cap B) - \chi(\mathbb{R}Z_1 \cap B) = \chi(\mathbb{R}Y_1) - \chi(\mathbb{R}Z_1),$$

and

$$\chi(\mathbb{R}Y_2 \cap B') - \chi(\mathbb{R}Z_2 \cap B') = \chi(\mathbb{R}Y_2) - \chi(\mathbb{R}Z_2).$$

So finally

$$\chi(\mathbb{R}Y_2) - \chi(\mathbb{R}Z_2) = \chi(\mathbb{R}Y_1) - \chi(\mathbb{R}Z_1).$$

It remains to compute  $\chi(\mathbb{R}Y_2)$  and  $\chi(\mathbb{R}Z_2)$ . Topologically,  $\mathbb{R}Z_2$  is obtained by taking in the quadrant  $++$  and  $+-$  (resp.,  $-+$  and  $--$ ) the “double” of  $(A \leq 0)$  (resp.,  $(A \geq 0)$ ) ramified along  $(A = 0) \cup (x = 0) \cup (x = \infty)$ . The same holds for  $\mathbb{R}Y_2$  by replacing  $A$  with  $\hat{A}$ , see Figure 13. By a direct computation, we obtain

$$\chi(\mathbb{R}Y_2) = 2(-18) - 12 = -48,$$

and

$$\chi(\mathbb{R}Z_2) = 2(-6) - 12 = -24.$$

Then,

$$\chi(\mathbb{R}Y) - \chi(\mathbb{R}Z) = \chi(\mathbb{R}Y_2) - \chi(\mathbb{R}Z_2) = -24.$$



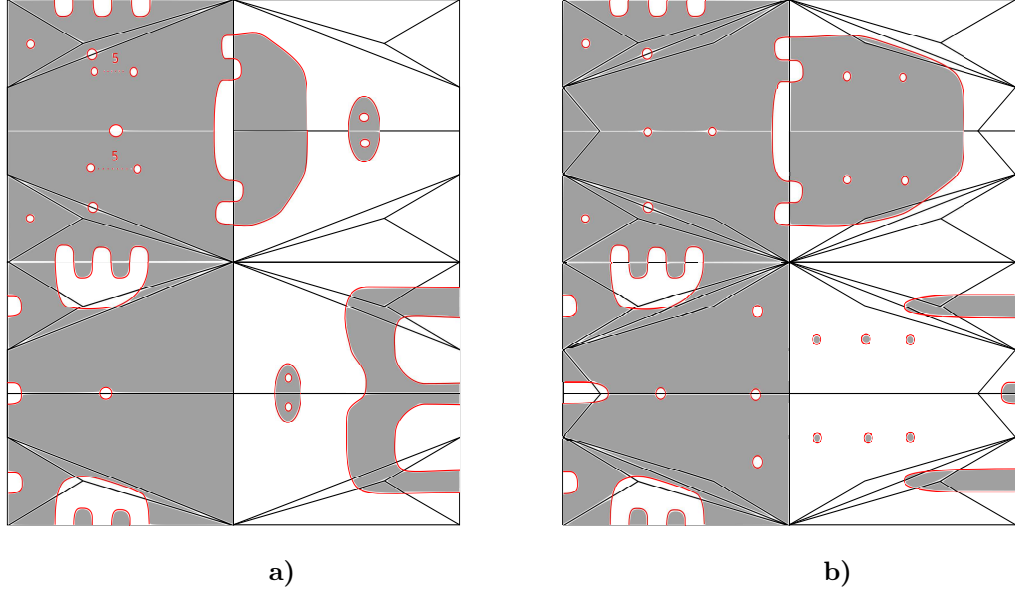


Figure 13: a):  $(A(x, y)x < 0)$     b):  $(\hat{A}(x, y)x < 0)$

So finally

$$\chi(\mathbb{R}Y) = \chi(\mathbb{R}Z) - 24 = -52 - 24 = -76.$$

Moreover,  $\mathbb{R}Y$  contains two components homeomorphic to  $S_2$  coming from the double covering of  $(\hat{A} > 0)$ . Note that the vertices  $(1, 1, 2)$ ,  $(1, 3, 1)$ ,  $(2, 3, 1)$  and  $(3, 3, 1)$  have the following property: all the vertices of the triangulation connected to one of these vertices by an edge have the sign  $+$ , while the vertices  $(1, 1, 2)$ ,  $(1, 3, 1)$ ,  $(2, 3, 1)$  and  $(3, 3, 1)$  have the sign  $-$ . Thus,  $\mathbb{R}Y$  contains also four spheres. There is at least one component of  $\mathbb{R}Y$  more: this component intersects the plane  $\{u = 0\}$ . Moreover,  $\mathbb{R}Y$  cannot have more components, otherwise  $Y$  would be an  $M$ -surface, but  $\chi(\mathbb{R}Y)$  does not satisfy the Rokhlin congruence. Finally, from  $\chi(\mathbb{R}Y) = -76$ , we obtain

$$\mathbb{R}Y \simeq 4S \amalg 2S_2 \amalg S_{41}.$$

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